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Character Analogues of the Poisson and Euler–MacLaurin Summation Formulas with Applications

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Character versions of the Poisson and Euler–Maclaurin summation formulas are derived. Instead of Bernoulli numbers and polynomials which appear in the classical Euler–Maclaurin formula, generalized Bernoulli numbers and polynomials now appear. Many applications of the results are given. In particular, applications are made to the evaluation of L-functions, the derivation of a character analogue of the Lipschitz summation formula, and the examination of a new character version of the classical gamma function.

1. INTRODUCTION

In 1904, Voronoï [25] conjectured that if $a(n)$ is a given arithmetical function, and f is a continuous function on $[a, b]$ with only a finite number of maxima and minima there, then there exist analytic functions $\alpha(x)$ and $\delta(x)$, depending only upon $a(n)$ and not upon $f(x)$, such that

$$\sum_{c\leqslant n\leqslant d}{}' a(n) f(n) = \int_c^d f(x)\, \delta(x)\, dx + \sum_{n=1}^{\infty} a(n) \int_c^d f(x)\, \alpha(nx)\, dx. \tag{1.1}$$

The prime on the summation sign on the left side of (1.1) means that if $n = c$ or $n = d$, only $\frac{1}{2}a(c) f(c)$ or $\frac{1}{2}a(d) f(d)$, respectively, is counted. In [9] we established for a class of arithmetical functions several general theorems which insure the validity of a somewhat more general form of (1.1). In [10] we considered briefly the special case when $a(n) = \chi(n)$, where χ is a primitive, nonprincipal character, and gave some applications. In this paper, we resume this study in detail. From our formulas, which

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can be regarded as character analogues of the Poisson summation formula, we shall derive character analogues of the Euler–Maclaurin summation formula.

Our study in [9] began with a general identity, due to Chandrasekharan and Narasimhan [12], for $\sum_{n\leqslant x} a(n)$ in terms of an infinite series of Bessel functions. This general result of Chandrasekharan and Narasimhan is quite deep and depends upon theorems in the theory of the uniform equiconvergence of trigonometric series. Since we are interested in only the special case $a(n) = \chi(n)$, it seems desirable to give an easier proof of the identity for $\sum_{n\leqslant x} \chi(n)$. This we do in Section 2. The method of proof is due to Hardy and Landau [18] and was used by them to obtain a corresponding result for $\sum_{n\leqslant x} r(n)$, where $r(n)$ is the number of representations of n as the sum of two squares. See also [19, pp. 221–232]. The method was also used by Hardy [17] in deriving an identity for $\sum_{n\leqslant x} \tau(n)$, where $\tau(n)$ denotes Ramanujan's arithmetical function. Finally, the method was used by us [6] in obtaining a like result for a class of arithmetical functions associated with entire modular forms.

In Section 3 we define generalized Bernoulli functions, polynomials and numbers. We shall show that each generalized Bernoulli function and polynomial is a certain linear combination of ordinary Bernoulli functions and polynomials, respectively.

In the next section we develop character versions of the Euler–Maclaurin summation formula. The Bernoulli functions and Bernoulli numbers which play a centeral role in the latter formula are replaced by generalized Bernoulli functions and generalized Bernoulli numbers in our new character analogues.

We shall give several applications of our character analogues of the Euler–Maclaurin formula in Section 5. In particular, we shall develop formulas for evaluating Dirichlet L-functions $L(n, \chi)$, where n is a suitable positive integer. We shall derive a character analogue of the Lipschitz summation formula. We also define and examine a new character analogue of the gamma function.

In the last section we indicate a few simple properties of generalized Bernoulli functions, polynomials and numbers. We shall also show that our definitions of generalized Bernoulli numbers and polynomials are equivalent to definitions given by Ankeny, Artin and Chowla [2] and Leopoldt [20] for real, primitive characters.

In the sequel, we shall always write $\sum$ for $\sum_{n=1}^{\infty}$. Also, A always denotes an unspecified positive number, not necessarily the same with each occurrence.

2. Character Analogues of the Poisson Summation Formula

Let χ be a primitive, nonprincipal character with modulus k. We define the Gaussian sums by

$$G(z, \chi) = \sum_{h=1}^{k-1} \chi(h)\, e^{2\pi i h z/k}.$$

Put $G(1, \chi) = G(\chi)$. If m is an integer, then we have the factorization theorem [5, p. 312],

$$G(m, \chi) = \bar{\chi}(m)\, G(\chi). \tag{2.1}$$

Let

$$L(s, \chi) = \sum \chi(n)\, n^{-s}, \qquad \sigma = \operatorname{Re} s > 0, \qquad b = \tfrac{1}{2}\{1 - \chi(-1)\},$$

and

$$\xi(s, \chi) = (\pi/k)^{-(s+b)/2}\, \Gamma(\tfrac{1}{2}\{s + b\})\, L(s, \chi).$$

Then $\xi(s, \chi)$ can be analytically continued to an entire function of s which satisfies the functional equation [5, p. 371]

$$\xi(s, \chi) = \epsilon(\chi)\, \xi(1 - s, \bar{\chi}), \tag{2.2}$$

where

$$\epsilon(\chi) = \begin{cases} k^{-1/2} G(\chi), & b = 0, \\ -ik^{-1/2} G(\chi), & b = 1. \end{cases}$$

We now state a special case of a general theorem of Chandrasekharan and Narasimhan [12, p. 6].

Theorem 2.1. *Let $\lambda_n = \pi n^2/k$. If $b = 0$, let $a(n) = \chi(n)$, $b(n) = \epsilon(\chi)\, \bar{\chi}(n)$, and $r = 1/2$. If $b = 1$, let $a(n) = n\chi(n)$, $b(n) = \epsilon(\chi)\, n\bar{\chi}(n)$, and $r = 3/2$. Then, for x, $q > 0$,*

$$\frac{1}{\Gamma(q+1)} \sum_{\lambda_n \leqslant x} a(n)(x - \lambda_n)^q = \sum b(n)(x/\lambda_n)^{(r+q)/2} J_{r+q}(2\{\lambda_n x\}^{1/2}),$$

where J_ν denotes the usual Bessel function of the first kind, and where the infinite series on the right side converges uniformly on any compact interval for $x > 0$.

The above result is quite easy to prove. However, the extension to $q = 0$ is somewhat difficult. Instead of quoting the theorem of Chandrasekharan and Narasimhan [12, p. 14], as remarked in the Introduction, we shall employ ideas of Hardy and Landau to give a simpler proof.

THEOREM 2.2. *In the notation of Theorem* 2.1, *for* $x > 0$,

$$\sum_{\lambda_n \leqslant x}{}' a(n) = \sum b(n)(x/\lambda_n)^{r/2} J_r(2\{\lambda_n x\}^{1/2}), \tag{2.3}$$

where the prime indicates that if $x = \lambda_n$, *only* $\frac{1}{2}a(n)$ *is counted. The infinite series on the right side of* (2.3) *converges boundedly on every compact interval in* $(0, \infty)$. *If the interval contains no members of* $\{\lambda_n\}$, *the convergence is uniform there.*

Proof. We need a few results about Bessel functions. For arbitrary ν and $c > 0$ [26, p. 45],

$$\frac{d}{dx}\{x^{-\nu/2} J_\nu(cx^{1/2})\} = -\tfrac{1}{2}cx^{-(\nu+1)/2} J_{\nu+1}(cx^{1/2}). \tag{2.4}$$

For arbitrary ν, as x tends to ∞ [26, p. 199],

$$J_\nu(x) = (2/\pi x)^{1/2}\cos(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(x^{-3/2}). \tag{2.5}$$

For $\nu > -1$ [26, p. 406],

$$\int_0^\infty J_\nu(ax)\, J_{\nu+1}(x)\, dx = \begin{cases} a^\nu, & 0 < a < 1, \\ \frac{1}{2}, & a = 1, \\ 0, & a > 1. \end{cases} \tag{2.6}$$

Now, in fact, we shall be considering only instances when $\nu \equiv 1/2 \pmod 1$. In such cases, $J_\nu(x)$ is a linear combination of certain powers of x each multiplied by either $\sin x$ or $\cos x$. Properties (2.4) and (2.5) are then trivial. However, we shall find it convenient to use the notation of Bessel functions. In particular, the cases $b = 0$ and $b = 1$ can be handled simultaneously.

For $q \geqslant 0$, let

$$B_q(x) = \sum_{\lambda_n \leqslant x} b(n)(x - \lambda_n)^q,$$

$$C_q = C_q(x, N) = B_q(N)(x/N)^{(r+q)/2} J_{r+q}(2\{Nx\}^{1/2}),$$

where $N > 0$, and

$$S = S(x, N) = \sum_{\lambda_n \leqslant N} b(n)(x/\lambda_n)^{r/2} J_r(2\{\lambda_n x\}^{1/2}).$$

Upon the use of (2.4) and an integration by parts,

$$\begin{aligned} S - C_0 &= -\sum_{\lambda_n \leqslant N} b(n) \int_{\lambda_n}^{N} d\{(x/u)^{r/2} J_r(2\{ux\}^{1/2})\} \\ &= \int_0^N B_0(u)(x/u)^{(r+1)/2} J_{r+1}(2\{ux\}^{1/2})\, du \\ &= C_1 + K, \end{aligned} \tag{2.7}$$

where

$$K = K(x, N) = \int_0^N B_1(u)(x/u)^{(r+2)/2} J_{r+2}(2\{ux\}^{1/2})\, du.$$

Now use Theorem 2.1 with $q = 1$ and the roles of $a(n)$ and $b(n)$ reversed to obtain

$$\begin{aligned} K &= \int_0^N (x/u)^{(r+2)/2} J_{r+2}(2\{ux\}^{1/2}) \sum a(n)(u/\lambda_n)^{(r+1)/2} J_{r+1}(2\{\lambda_n u\}^{1/2})\, du \\ &= x^{1/2} \sum a(n)(x/\lambda_n)^{(r+1)/2} \int_0^N u^{-1/2} J_{r+1}(2\{\lambda_n u\}^{1/2})\, J_{r+2}(2\{ux\}^{1/2})\, du \\ &= \sum a(n)(x/\lambda_n)^{(r+1)/2} \int_0^{2\{Nx\}^{1/2}} J_{r+1}(u\{\lambda_n/x\}^{1/2})\, J_{r+2}(u)\, du, \end{aligned}$$

where the inversion in order of summation and integration is justified by absolute convergence. If we denote the left side of (2.3) by $A(x)$ and use (2.6), we find that

$$\begin{aligned} K &= \sum a(n)(x/\lambda_n)^{(r+1)/2} \left\{\int_0^\infty - \int_{2\{Nx\}^{1/2}}^\infty\right\} J_{r+1}(u\{\lambda_n/x\}^{1/2})\, J_{r+2}(u)\, du \\ &= A(x) - L, \end{aligned}$$

where

$$L = L(x, N) = \sum a(n)(x/\lambda_n)^{(r+1)/2} \int_{2\{Nx\}^{1/2}}^\infty J_{r+1}(u\{\lambda_n/x\}^{1/2})\, J_{r+2}(u)\, du.$$

Thus, from (2.7) we have

$$S - A(x) = C_0 + C_1 - L. \tag{2.8}$$

Assume that $0 < x_1 \leqslant x \leqslant x_2$. We must show that the right side of (2.8) tends to 0 boundedly on $[x_1, x_2]$ as N tends to ∞. Moreover, if $[x_1, x_2]$ is free of members of $\{\lambda_n\}$, we must show that the right side of (2.8) tends to 0 uniformly on $[x_1, x_2]$.

First, from Theorem 2.1 and (2.5), we easily deduce that

$$| B_1(N)| \leqslant AN^{r/2+1/4} \sum | a(n)| \lambda_n^{-r/2-3/4}. \tag{2.9}$$

In both cases, we note that the series on the right side of (2.9) converges. Thus, using (2.5) and (2.9), we find that

$$C_1(x, N) = O(N^{-1/2}) \tag{2.10}$$

as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$.

In the examination of C_0, we shall consider the cases of even χ and odd χ separately. Since

$$\sum_{n=1}^{k-1} \chi(n) = 0, \tag{2.11}$$

it follows trivially that for even χ, $B_0(N) = O(1)$ as N tends to ∞. Thus, using (2.5), we deduce that for even χ,

$$C_0(x, N) = O(N^{-1/2}) \tag{2.12}$$

as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$. Now let χ be odd. Noting that $B_0(N)$ contains $O(N^{1/2})$ terms, it is an easy exercise in partial summation to show that $B_1(N) = O(N^{1/2})$ as N tends to ∞. Thus, from (2.5), we find that for odd χ,

$$C_0(x, N) = O(N^{-1/2}) \tag{2.13}$$

as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$.

We now examine L. From (2.5), we see that as u tends to ∞,

$$\begin{aligned} &J_{r+1}(u\{\lambda_n/x\}^{1/2})\, J_{r+2}(u) \\ &\quad = -2(x/\lambda_n)^{1/4} \cos(u\{\lambda_n/x\}^{1/2} - \tfrac{1}{2}r\pi - 3\pi/4) \cos(u - \tfrac{1}{2}r\pi - \pi/4)/\pi u \\ &\qquad + O(u^{-2}\lambda_n^{-3/4}x^{3/4}) + O(u^{-2}\lambda_n^{-1/4}x^{1/4}) + O(u^{-3}\lambda_n^{-3/4}x^{3/4}) \\ &\quad = Q(u) + O(u^{-2}\lambda_n^{-1/4}), \end{aligned} \tag{2.14}$$

uniformly for $0 < x_1 \leqslant x \leqslant x_2$, where

$$Q(u) = -2(x/\lambda_n)^{1/4} \cos(u\{\lambda_n/x\}^{1/2} - \tfrac{1}{2}r\pi - 3\pi/4) \cos(u - \tfrac{1}{2}r\pi - \pi/4)/\pi u.$$

The contribution to L of the big O term on the far right side of (2.14) is

$$O\left(\sum | a(n)| \lambda_n^{-r/2-3/4} \int_{2\{Nx\}^{1/2}}^{\infty} u^{-2}\, du\right) = O(N^{-1/2})$$

as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$. The contribution of $Q(u)$ to L is

$$\frac{1}{\pi} \sum a(n)(x/\lambda_n)^{r/2+3/4} \left\{ \int_{2\{Nx\}^{1/2}}^{\infty} u^{-1} \cos\{u(1 + \{\lambda_n/x\}^{1/2}) - r\pi\}\, du \right.$$

$$\left. + \int_{2\{Nx\}^{1/2}}^{\infty} u^{-1} \sin\{u(1 - \{\lambda_n/x\}^{1/2})\}\, du \right\} = S_1 + S_2,$$

say. By an integration by parts, each of the integrals in S_1 is easily seen to be $O(N^{-1/2})$ as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$. Thus,

$$| S_1 | \leqslant AN^{-1/2} \sum | a(n)| \, (x/\lambda_n)^{r/2+3/4} = O(N^{-1/2}) \tag{2.15}$$

as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$. In S_2, let $v = u \mid 1 - \{\lambda_n/x\}^{1/2} \mid$. Then,

$$S_2 = \frac{1}{\pi} \sum a(n)(x/\lambda_n)^{r/2+3/4} \operatorname{sgn}(1 - \{\lambda_n/x\}^{1/2}) \int_{2\{Nx\}^{1/2}|1-\{\lambda_n/x\}^{1/2}|}^{\infty} v^{-1} \sin v \, dv.$$

If no member of $\{\lambda_n\}$ is contained in $[x_1, x_2]$, we see by an integration by parts that $S_2 = O(N^{-1/2})$ as N tends to ∞, uniformly for $0 < x_1 \leqslant x \leqslant x_2$. In general, for each fixed x, $0 < x_1 \leqslant x \leqslant x_2$, either $\operatorname{sgn}(1 - \{\lambda_n/x\}^{1/2}) = 0$ or the integral is $O(N^{-1/2})$ as N tends to ∞. Thus, $S_2 = O(N^{-1/2})$ as N tends to ∞, for each fixed x. However, the integrals in S_2 are uniformly bounded for $0 < x_1 \leqslant x \leqslant x_2$, $n \geqslant 1$, and $N \geqslant 1$. Thus, S_2 tends to 0 boundedly on $0 < x_1 \leqslant x \leqslant x_2$ as N tends to ∞. Combining this result with (2.10), (2.12), (2.13) and (2.15), we see from (2.8) that $S(x, N)$ tends to $A(x)$ uniformly on $[x_1, x_2]$ if the interval is free of members of $\{\lambda_n\}$. In general, $S(x, N)$ converges boundedly to $A(x)$ on $[x_1, x_2]$. This completes the proof of Theorem 2.2.

To obtain character analogues of the Poisson summation formula, we may now apply any of the main results in [9]. Thus, if $f \in C^{(1)}[0, x]$, we have the following two results stated in [10]. If χ is even [10, Eq. (6.3)],

$$\sum_{n \leqslant x}{}' \chi(n) f(\pi n^2/k)$$

$$= 2G(\chi)(\pi k)^{-1/2} \sum \bar{\chi}(n) \int_0^{x(\pi/k)^{1/2}} f(u^2) \cos(2nu\{\pi/k\}^{1/2})\, du; \tag{2.16}$$

if χ is odd [10, Eq. (6.4)],

$$\sum_{n \leqslant x}' n\chi(n) f(\pi n^2/k)$$
$$= -2iG(\chi)\, \pi^{-1} \sum \bar{\chi}(n) \int_0^{x(\pi/k)^{1/2}} uf(u^2) \sin(2nu\{\pi/k\}^{1/2})\, du. \quad (2.17)$$

In (2.16) replace $f(\pi n^2/k)$ by $f(n)$, and in (2.17) replace $nf(\pi n^2/k)$ by $f(n)$. Replace $u(k/\pi)^{1/2}$ by u and the interval $[0, x]$ by $[a, b]$, where $0 \leqslant a < b < \infty$. We may also weaken the hypotheses by assuming that f is of bounded variation on $[a, b]$ [9, Theorems 6 and 9]. It is a simple exercise to show that the conditions $a, b \geqslant 0$ are unnecessary; we may assume $-\infty < a < b < \infty$. Hence, we have the following.

THEOREM 2.3. *Let f be of bounded variation on $[a, b]$, $-\infty < a < b < \infty$. If χ is even,*

$$\tfrac{1}{2} \sum_{a \leqslant n \leqslant b}' \chi(n)\{f(n+0) + f(n-0)\} = \frac{2G(\chi)}{k} \sum \bar{\chi}(n) \int_a^b f(u) \cos(2\pi nu/k)\, du; \quad (2.18)$$

if χ is odd,

$$\tfrac{1}{2} \sum_{a \leqslant n \leqslant b}' \chi(n)\{f(n+0) + f(n-0)\}$$
$$= \frac{-2iG(\chi)}{k} \sum \bar{\chi}(n) \int_a^b f(u) \sin(2\pi nu/k)\, du. \quad (2.19)$$

A. P. Guinand [15, 16] has also derived character analogues of Poisson's formula, but under quite different hypotheses.

3. GENERALIZED BERNOULLI FUNCTIONS, POLYNOMIALS AND NUMBERS

Let $B_j(x)$ denote the jth Bernoulli polynomial. Let $\mathscr{B}_j(x)$ be defined by

$$\mathscr{B}_j(x + n) = B_j(x),$$

where n is an arbitrary integer and $0 \leqslant x < 1$ if $j \geqslant 2$ and $0 < x < 1$ if $j = 1$. Define $\mathscr{B}_1(n) = 0$ for every integer n. Then for $-\infty < x < \infty$ we have the following Fourier series expansions [1, p. 805],

$$\mathscr{B}_{2j-1}(x) = \frac{2(-1)^j (2j-1)!}{(2\pi)^{2j-1}} \sum \frac{\sin(2\pi nx)}{n^{2j-1}} \quad (3.1)$$

and

$$\mathscr{B}_{2j}(x) = \frac{2(-1)^{j-1}(2j)!}{(2\pi)^{2j}} \sum \frac{\cos(2\pi nx)}{n^{2j}}. \tag{3.2}$$

DEFINITION 1. The generalized Bernoulli functions $\mathscr{B}_j(x, \chi)$ are defined as follows. Let $-\infty < x < \infty$ and $j \geqslant 1$. If χ is even,

$$\mathscr{B}_{2j-1}(x, \chi) = \frac{2(-1)^j G(\bar{\chi})(2j-1)!}{k(2\pi/k)^{2j-1}} \sum \frac{\chi(n)\sin(2\pi nx/k)}{n^{2j-1}}$$

and

$$\mathscr{B}_{2j}(x, \chi) = \frac{2(-1)^{j-1} G(\bar{\chi})(2j)!}{k(2\pi/k)^{2j}} \sum \frac{\chi(n)\cos(2\pi nx/k)}{n^{2j}};$$

if χ is odd,

$$\mathscr{B}_{2j-1}(x, \chi) = \frac{2(-1)^{j-1} iG(\bar{\chi})(2j-1)!}{k(2\pi/k)^{2j-1}} \sum \frac{\chi(n)\cos(2\pi nx/k)}{n^{2j-1}}$$

and

$$\mathscr{B}_{2j}(x, \chi) = \frac{2(-1)^{j-1} iG(\bar{\chi})(2j)!}{k(2\pi/k)^{2j}} \sum \frac{\chi(n)\sin(2\pi nx/k)}{n^{2j}}.$$

THEOREM 3.1. *For $j \geqslant 1$ and $-\infty < x < \infty$,*

$$\mathscr{B}_j(x, \chi) = k^{j-1} \sum_{h=1}^{k-1} \bar{\chi}(h)\, \mathscr{B}_j\left(\frac{x+h}{k}\right).$$

Proof. We shall prove the theorem in the case when χ is even and j is odd. The proofs in the remaining three cases are completely analogous. By (2.1),

$$\sum \frac{\chi(n)\sin(2\pi nx/k)}{n^{2j-1}} = \sum \frac{G(n, \bar{\chi})\sin(2\pi nx/k)}{G(\bar{\chi})\, n^{2j-1}}$$

$$= \frac{1}{G(\bar{\chi})} \sum \frac{1}{n^{2j-1}} \sum_{h=1}^{k-1} \bar{\chi}(h)\, e^{2\pi ihn/k} \sin(2\pi nx/k)$$

$$= \frac{1}{2G(\bar{\chi})} \sum \frac{1}{n^{2j-1}} \sum_{h=1}^{k-1} \bar{\chi}(h)\{\sin(2\pi n[x+h]/k) + \sin(2\pi n[x-h]/k)$$

$$- i\cos(2\pi n[x+h]/k) + i\cos(2\pi n[x-h]/k)\}. \tag{3.3}$$

We now replace h be $k - h$ and use the facts that χ is even and has period k to obtain

$$\sum_{h=1}^{k-1} \bar{\chi}(h) \cos(2\pi n[x - h]/k) = \sum_{h=1}^{k-1} \bar{\chi}(h) \cos(2\pi n[x + h]/k). \tag{3.4}$$

Similarly,

$$\sum_{h=1}^{k-1} \bar{\chi}(h) \sin(2\pi n[x - h]/k) = \sum_{h=1}^{k-1} \bar{\chi}(h) \sin(2\pi n[x + h]/k). \tag{3.5}$$

If we substitute (3.4) and (3.5) into the right side of (3.3), invert the order of summation, and use (3.1), we find that

$$\begin{aligned}\sum \frac{\chi(n) \sin(2\pi nx/k)}{n^{2j-1}} &= \frac{1}{G(\bar{\chi})} \sum_{h=1}^{k-1} \bar{\chi}(h) \sum \frac{\sin(2\pi n[x + h]/k)}{n^{2j-1}} \\ &= \frac{(-1)^j (2\pi)^{2j-1}}{2G(\bar{\chi})(2j-1)!} \sum_{h=1}^{k-1} \bar{\chi}(h) \mathscr{B}_{2j-1}\left(\frac{x+h}{k}\right).\end{aligned}$$

Upon using Definition 1, we find that the proof is complete.

DEFINITION 2. The generalized Bernoulli polynomials $B_j(z, \chi)$, $j \geqslant 1$, are defined as follows. For $0 < x < 1$,

$$B_j(x, \chi) = \mathscr{B}_j(x, \chi).$$

$B_j(z, \chi)$ is defined for all complex z by analytic continuation. By Theorem 3.1, $B_j(z, \chi)$ is, indeed, a polynomial in z.

The next result follows immediately from Theorem 3.1 and Definition 2.

COROLLARY 3.2. *For $j \geqslant 1$,*

$$B_j(z, \chi) = k^{j-1} \sum_{h=1}^{k-1} \bar{\chi}(h) B_j\left(\frac{z+h}{k}\right).$$

From Theorem 3.1, (2.11), and the fact that $B_j(z)$ is of degree j, we observe that the degree of $B_j(z, \chi)$ is no greater than $j - 1$.

COROLLARY 3.3. *For $j \geqslant 2$,*

$$\mathscr{B}_j'(z, \chi) = j\mathscr{B}_{j-1}(z, \chi).$$

Proof. The result follows immediately from the corresponding result for ordinary Bernoulli polynomials [1, p. 804],

$$B_j'(z) = jB_{j-1}(z),$$

and Corollary 3.2. Alternatively, the result is a consequence of Definition 1.

DEFINITION 3. The generalized Bernoulli numbers $B_j(\chi)$, $j \geqslant 1$, are defined by

$$B_j(\chi) = B_j(0, \chi).$$

Note that the definition of the generalized Bernoulli numbers is completely analogous to that of the ordinary Bernoulli numbers. The notation $B_j(\chi)$ conflicts slightly with that of the Bernoulli polynomial $B_j(x)$, but there will be no cause for confusion in the sequel. Observe that $B_j(x, \bar{\chi}) = \chi(-1)\,\overline{B_j(x, \chi)}$ and $B_j(\bar{\chi}) = \chi(-1)\,\overline{B_j(\chi)}$.

From Definitions 1–3 we immediately obtain the following.

COROLLARY 3.4. *If χ is even, $B_{2j-1}(\chi) = 0$; if χ is odd, $B_{2j}(\chi) = 0$.*

In the sequel we shall frequently be examining integrals involving the Fourier series of generalized Bernoulli functions. Except for $\mathscr{B}_1(x, \chi)$, the inversion of order of summation and integration will always be justified by absolute convergence. It is well known that the Fourier series of $\mathscr{B}_1(x)$ is boundedly convergent. Hence, by Theorem 3.1, the Fourier series of $\mathscr{B}_1(x, \chi)$ is boundedly convergent. Thus, the interchange in order of summation and integration over a finite interval is justified [23, p. 41] for the case of $\mathscr{B}_1(x, \chi)$ as well.

4. CHARACTER ANALOGUES OF THE EULER–MACLAURIN SUMMATION FORMULA

THEOREM 4.1. *Let $f \in C^{(m+1)}[a, b]$, $-\infty < a < b < \infty$. If χ is even,*

$$\begin{aligned}\sum_{a \leqslant n \leqslant b}' \chi(n) f(n) = &\sum_{j=0}^{m} \frac{\mathscr{B}_{j+1}(b, \bar{\chi})}{(j+1)!} (-1)^{j+1} f^{(j)}(b) \\ &+ \sum_{j=0}^{m} \frac{\mathscr{B}_{j+1}(a, \bar{\chi})}{(j+1)!} (-1)^{j} f^{(j)}(a) \\ &+ \frac{(-1)^m}{(m+1)!} \int_a^b \mathscr{B}_{m+1}(u, \bar{\chi}) f^{(m+1)}(u)\, du; \qquad (4.1)\end{aligned}$$

if χ is odd,

$$\sum_{a\leqslant n\leqslant b}{}' \chi(n) f(n) = \sum_{j=0}^{m} \frac{\mathscr{B}_{j+1}(b, \bar{\chi})}{(j+1)!} (-1)^j f^{(j)}(b)$$

$$+ \sum_{j=0}^{m} \frac{\mathscr{B}_{j+1}(a, \bar{\chi})}{(j+1)!} (-1)^{j+1} f^{(j)}(a)$$

$$+ \frac{(-1)^{m+1}}{(m+1)!} \int_a^b \mathscr{B}_{m+1}(u, \bar{\chi}) f^{(m+1)}(u)\, du. \qquad (4.2)$$

In particular, let $a = Ak$ and $b = Bk$, where A and B are integers. If $m = 2r$ is even, (4.1) *reduces to*

$$\sum_{Ak\leqslant n\leqslant Bk} \chi(n) f(n) = \sum_{j=1}^{r} \frac{B_{2j}(\bar{\chi})}{(2j)!} \{f^{(2j-1)}(Bk) - f^{(2j-1)}(Ak)\}$$

$$+ \frac{1}{(2r+1)!} \int_{Ak}^{Bk} \mathscr{B}_{2r+1}(u, \bar{\chi}) f^{(2r+1)}(u)\, du; \qquad (4.3)$$

if $m = 2r - 1$ is odd, (4.2) *reduces to*

$$\sum_{Ak\leqslant n\leqslant Bk} \chi(n) f(n) = \sum_{j=1}^{r} \frac{B_{2j-1}(\bar{\chi})}{(2j-1)!} \{f^{(2j-2)}(Bk) - f^{(2j-2)}(Ak)\}$$

$$+ \frac{1}{(2r)!} \int_{Ak}^{Bk} \mathscr{B}_{2r}(u, \bar{\chi}) f^{(2r)}(u)\, du. \qquad (4.4)$$

Proof. Let χ be even and consider (2.18). Upon an integration by parts, we find with the aid of Definition 1 that

$$\sum_{a\leqslant n\leqslant b}{}' \chi(n) f(n) = -\mathscr{B}_1(b, \bar{\chi}) f(b) + \mathscr{B}_1(a, \bar{\chi}) f(a) + \int_a^b \mathscr{B}_1(u, \bar{\chi}) f'(u)\, du.$$

Upon m additional integrations by parts with the help of Corollary 3.3, we deduce (4.1).

The proof of (4.2) is analogous and uses (2.19) rather than (2.18).

Equations (4.3) and (4.4) follow readily from (4.1) and (4.2), respectively, upon the use of Definition 3 and Corollary 3.4.

J. B. Rosser and L. Schoenfeld [22] have also derived a version of Theorem 4.1.

THEOREM 4.2. *Let $f \in C^{(\infty)}[0, \infty)$. Suppose that $f^{(j)}(x)$, $j \geqslant 0$, tends to* 0 *as x tends to ∞. Assume that*

$$\frac{1}{r!} \int_0^\infty \mathscr{B}_r(u, \bar{\chi}) f^{(r)}(u)\, du$$

tends to 0 *as r tends to* ∞. *Then if* χ *is even,*

$$\sum \chi(n) f(n) = -\sum \frac{B_{2j}(\bar{\chi})}{(2j)!} f^{(2j-1)}(0); \tag{4.5}$$

if χ *is odd,*

$$\sum \chi(n) f(n) = -\sum \frac{B_{2j-1}(\bar{\chi})}{(2j-1)!} f^{(2j-2)}(0). \tag{4.6}$$

Proof. The results easily follow from (4.3) and (4.4), respectively.

Theorem 4.2 was conjectured for real primitive characters by Chowla [13] who gave an application of his conjecture in the case $\chi(n) = (-4 \mid n)$, where $(m \mid n)$ denotes the Kronecker symbol. In this case,

$$\begin{aligned} B_j(x, \chi) &= 4^{j-1} \sum_{h=1}^{3} (-4 \mid h)\, B_j(\tfrac{1}{4}\{x + h\}) \\ &= 4^{j-1}\{B_j(\tfrac{1}{4}\{x + 1\}) - B_j(\tfrac{1}{4}\{x + 3\})\}. \end{aligned}$$

Now, if $E_j(x)$ denotes the jth Euler polynomial [1, p. 806],

$$E_{j-1}(x) = (2^j/j)\{B_j(\tfrac{1}{2}\{x + 1\}) - B_j(\tfrac{1}{2}x)\}.$$

Thus,

$$B_j(x, \chi) = -2^{j-2} j E_{j-1}(\tfrac{1}{2}\{x + 1\}).$$

The Euler numbers E_j are defined by [1, p. 804]

$$E_j = 2^j E_j(\tfrac{1}{2}).$$

Thus,

$$B_j(\chi) = -jE_{j-1}/4,$$

and (4.4) reduces to

$$\begin{aligned} \sum_{4A \leqslant 2n+1 \leqslant 4B} (-1)^n f(2n + 1) &= \frac{1}{4} \sum_{j=0}^{r-1} \frac{E_{2j}}{(2j)!} \{f^{(2j)}(4A) - f^{(2j)}(4B)\} \\ &\quad - \frac{2^{2r-2}}{(2r-1)!} \int_{4A}^{4B} \mathscr{E}_{2r-1}(\tfrac{1}{2}\{u + 1\})\, f^{(2r)}(u)\, du, \end{aligned}$$

where for any integer n,

$$\mathscr{E}_j(\tfrac{1}{2}\{x + 4n + 1\}) = E_j(\tfrac{1}{2}\{x + 1\}), \qquad 0 \leqslant x < 4.$$

Results (4.3)–(4.6) can be written in terms of values of Dirichlet

L-functions. Indeed, suppose that χ is even. From Definition 1 we find immediately that for $j \geqslant 1$,

$$B_{2j}(\chi) = \frac{2(-1)^{j-1}\, G(\bar{\chi})(2j)!}{k(2\pi/k)^{2j}}\, L(2j, \chi). \tag{4.7}$$

If χ is odd, for $j \geqslant 1$,

$$B_{2j-1}(\chi) = \frac{2(-1)^{j-1}\, iG(\bar{\chi})(2j-1)!}{k(2\pi/k)^{2j-1}}\, L(2j-1, \chi). \tag{4.8}$$

Hence, if $n \equiv b \pmod 2$, we have a simple formula for $L(n, \chi)$, $n \geqslant 1$, in terms of generalized Bernoulli numbers. If χ is even, we find from (2.2) that if j is a positive integer,

$$L(2j, \chi) = \tfrac{1}{2}(-1)^j\, G(\chi)(2\pi/k)^{2j}\, L(1-2j, \bar{\chi})/\Gamma(2j).$$

Upon substitution in (4.7), we obtain

$$B_{2j}(\chi) = -2jL(1-2j, \bar{\chi}),$$

where we have used the fact that for an arbitrary primitive character [5, p. 313],

$$|\, G(\chi)|^2 = k. \tag{4.9}$$

A similar result holds if χ is odd.

5. Applications

Example 1. In the derivation of Theorem 2.3, the primitivity of χ was used only once, namely in order to prove the functional equation (2.2). Suppose now that we make no assumption on the primitivity of χ, but let us assume that $L(s, \chi)$ satisfies (2.2). Let $f(x) = \exp(2\pi imx/k)$ in (2.18) or (2.19), depending upon whether χ is even or odd, where m is a positive integer. In either case, we obtain at once

$$G(m, \chi) = \bar{\chi}(m)\, G(\chi), \tag{5.1}$$

in other words, the factorization theorem (2.1). Apostol [3] has shown that if (5.1) holds, then χ is necessarily primitive. Hence, we have shown that if $L(s, \chi)$ satisfies the functional equation (2.2), then χ must be primitive. This result has also recently been proved by Apostol [4] in a different way.

EXAMPLE 2. First, define

$$M_m(\chi) = \sum_{n=1}^{k-1} \chi(n)\, n^m.$$

Next, from either Definition 1 or Corollary 3.3, it is plain that

$$\int_0^k \mathscr{B}_j(u, \chi)\, du = 0. \tag{5.2}$$

Let χ be even. Put $f(x) = x^m$, where m is a positive integer, $A = 0$, $B = 1$, and $r = [m/2]$ in (4.3). Using (5.2) and (4.7), we find that

$$\begin{aligned} M_m(\chi) &= \sum_{j=1}^{[m/2]} \frac{B_{2j}(\bar{\chi})\, m!\, k^{m-2j+1}}{(2j)!\,(m-2j+1)!} \\ &= 2G(\chi)\, k^m \sum_{j=1}^{[m/2]} (2\pi)^{-2j} (-1)^{j-1} \frac{m!}{(m-2j+1)!} L(2j, \bar{\chi}). \end{aligned} \tag{5.3}$$

In [10] we obtained (5.3) using Theorem 2.3 instead of Theorem 4.1. In the latter use, the calculation is easier. Equation (5.3) enables us to find recursively exact formulas for $L(2n, \chi)$, $n \geqslant 1$. For example, letting $m = 2$ and using (4.9), we obtain

$$L(2, \chi) = (\pi^2/k^3)\, G(\chi)\, M_2(\bar{\chi}).$$

Let χ be odd and f be as above. Put $A = 0$, $B = 1$, and $r = [(m+1)/2]$ in (4.4). Using (5.2) and (4.8), we obtain

$$M_m(\chi) = 2iG(\chi)\, k^m \sum_{j=1}^{[(m+1)/2]} (2\pi)^{-2j+1} (-1)^{j-1} \frac{m!}{(m-2j+2)!} L(2j-1, \bar{\chi}). \tag{5.4}$$

Equation (5.4) may be used recursively to find exact formulas for $L(2n-1, \chi)$, $n \geqslant 1$. Thus, letting $m = 1$, for example, we have with the help of (4.9),

$$L(1, \chi) = (i\pi/k^2)\, G(\chi)\, M_1(\bar{\chi}).$$

EXAMPLE 3. We shall derive a character analogue of the Lipschitz summation formula [21]. For $\operatorname{Re} z > 0$, $\sigma > 1$ and α real, let $f(u) = (z + ui)^{-s} \exp(2\pi i u\alpha/k)$. For χ even, we deduce from (2.18) that

$$\begin{aligned}
&\sum_{a \leqslant n \leqslant b}' \chi(n) \frac{e^{2\pi i n\alpha/k}}{(z+ni)^s} \\
&\quad = \frac{2G(\chi)}{k} \sum \bar{\chi}(n) \int_a^b \frac{e^{2\pi i u\alpha/k}}{(z+ui)^s} \cos(2\pi n u/k)\, du \\
&\quad = \frac{2G(\chi)}{k} \sum \bar{\chi}(n) \left\{ \int_{-\infty}^{\infty} - \int_{-\infty}^{a} - \int_b^{\infty} \right\} \frac{e^{2\pi i u\alpha/k}}{(z+ui)^s} \cos(2\pi n u/k)\, du. \qquad (5.5)
\end{aligned}$$

Observe that the infinite integral above converges for $\sigma > 1$. We wish to let a tend to $-\infty$ and b to $+\infty$ on the right side of (5.5). Suppose that $b = Nk$, where N is a positive integer. Integrating by parts twice, we get

$$\begin{aligned}
&\sum_{n+\alpha\neq 0} \bar{\chi}(n) \int_b^{\infty} \frac{e^{2\pi i u(\alpha+n)/k}}{(z+ui)^s}\, du \\
&\quad = -\frac{k e^{2\pi i N\alpha}}{2\pi i (z+Nki)^s} \sum_{n+\alpha\neq 0} \frac{\bar{\chi}(n)}{n+\alpha} + \frac{sk}{2\pi i} \sum_{n+\alpha\neq 0} \frac{\bar{\chi}(n)}{n+\alpha} \int_b^{\infty} \frac{e^{2\pi i u(\alpha+n)/k}}{(z+ui)^{s+1}}\, du \\
&\quad = -\frac{k e^{2\pi i N\alpha}}{2\pi i (z+Nki)^s} \sum_{n+\alpha\neq 0} \frac{\bar{\chi}(n)}{n+\alpha} + O\left((z+Nki)^{-\sigma-1} \sum n^{-2} \right),
\end{aligned}$$

which tends to 0 as N tends to ∞, since $\sigma > 1$. A similar result holds for the integral involving $\exp(2\pi i u(\alpha - n)/k)$ as well as the integrals over $(-\infty, a)$ as a tends to $-\infty$. Thus, letting a tend to $-\infty$ and b tend to $+\infty$ in (5.5), we arrive at for $\sigma > 1$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \chi(n) \frac{e^{2\pi i n\alpha/k}}{(z+ni)^s} &= \frac{2G(\chi)}{k} \sum \bar{\chi}(n) \int_{-\infty}^{\infty} \frac{e^{2\pi i u\alpha/k}}{(z+ui)^s} \cos(2\pi n u/k)\, du \\
&= \frac{G(\chi)}{k^s} \sum \bar{\chi}(n) \int_{-\infty}^{\infty} \frac{e^{2\pi i u(\alpha+n)} + e^{2\pi i u(\alpha-n)}}{(z/k+ui)^s}\, du \\
&= \frac{G(\chi)}{k^s} \sum_{n=-\infty}^{\infty} \bar{\chi}(n) \int_{-\infty}^{\infty} \frac{e^{2\pi i u(n+\alpha)}}{(z/k+ui)^s}\, du, \qquad (5.6)
\end{aligned}$$

since χ is even. Using the well-known result,

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i u(n+\alpha)}}{(z/k+ui)^s}\, du = \begin{cases} 0, & n+\alpha \leqslant 0, \\ \dfrac{(2\pi)^s}{\Gamma(s)} (n+\alpha)^{s-1} e^{-2\pi z(n+\alpha)/k}, & n+\alpha > 0, \end{cases}$$

where $\sigma > 1$, we find that (5.6) becomes for $\sigma > 1$

$$\sum_{n=-\infty}^{\infty} \chi(n) \frac{e^{2\pi i n\alpha/k}}{(z+ni)^s} = \frac{(2\pi/k)^s\, G(\chi)}{\Gamma(s)} \sum_{n+\alpha>0} \bar{\chi}(n)(n+\alpha)^{s-1}\, e^{-2\pi z(n+\alpha)/k}.$$

If χ is odd, we may use (2.19) and proceed in a fashion similar to that above. In summary, for any primitive character χ, $\operatorname{Re} z > 0$, α real, and $\sigma > 1$, we have

$$\sum_{n=-\infty}^{\infty} \chi(n) \frac{e^{2\pi i n\alpha/k}}{(z+ni)^s} = \chi(-1) \frac{(2\pi/k)^s\, G(\chi)}{\Gamma(s)} \sum_{n+\alpha>0} \bar{\chi}(n)(n+\alpha)^{s-1}\, e^{-2\pi z(n+\alpha)/k}. \tag{5.7}$$

We now wish to show that in most cases (5.7) is valid for $\sigma > 0$. For fixed z, $\operatorname{Re} z > 0$, the right side of (5.7) can clearly be analytically continued to an entire function of s. Suppose first that α is not an integer. Letting $n = mk + r$, $0 \leqslant m \leqslant N-1$, $0 \leqslant r \leqslant k-1$, we obtain

$$\left| \sum_{n=1}^{Nk} \chi(n)\, e^{2\pi i n\alpha/k} \right| = \left| \sum_{r=0}^{k-1} \chi(r)\, e^{2\pi i r\alpha/k} \sum_{m=0}^{N-1} e^{2\pi i m\alpha} \right| \leqslant 2k \mid 1 - e^{2\pi i\alpha} \mid^{-1} = O(1),$$

as N tends to ∞. Hence, the Dirichlet series on the left side of (5.7) converges for $\sigma > 0$, and from the general theory of Dirichlet series, is analytic for $\sigma > 0$. Thus, if α is not an integer, (5.7) is valid by analytic continuation for $\sigma > 0$. Next, let α be an integer. Then by the previous calculation and (2.1),

$$\sum_{n=1}^{Nk} \chi(n)\, e^{2\pi i n\alpha/k} = NG(\alpha, \chi) = N\bar{\chi}(\alpha)\, G(\chi).$$

If $(\alpha, k) > 1$, the above is 0. Hence, the Dirichlet series on the left side of (5.7) is analytic for $\sigma > 0$, and (5.7) is valid for $\sigma > 0$ by analytic continuation. If $(\alpha, k) = 1$, the abscissa of convergence of the Dirichlet series on the left side of (5.7) is one, and no continuation is possible.

Example 4. The functional equation of the Riemann zeta-function may be derived from the Euler–Maclaurin sum formula [24, pp. 14–15].

We shall show here that in a similar fashion, the functional equation of a class of generalized L-functions may be derived from Theorem 4.1. A special case of our result is a character analogue of Hurwitz's formula for $\zeta(s, a)$, the Hurwitz zeta-function.

For x and a real, define

$$L(s, x, a, \chi) = \sum_{n=0}^{\infty}{}' e^{2\pi inx/k}\chi(n)(n+a)^{-s},$$

where the prime indicates that the term corresponding to $n = -a$ is omitted if a is a negative integer and $\chi(a) \neq 0$. From the remarks made at the end of Example 3, the series above converges for $\sigma > 0$ if x is not an integer, or if x is an integer and $(x, k) > 1$. If x is an integer and $(x, k) = 1$, the series converges for $\sigma > 1$.

THEOREM 5.1. *Let x be real and $0 \leqslant a \leqslant 1$. Then $L(s, x, a, \chi)$ can be analytically continued to an entire function of s except when x is integral and $(x, k) = 1$, in which case $L(s, x, a, \chi)$ is analytic everywhere except at $s = 1$, where there is a simple pole with residue $G(x, \chi)/k$. Furthermore, if $0 \leqslant x \leqslant 1$, $L(s, x, a, \chi)$ satisfies the functional equation,*

$$L(1-s, x, a, \chi) = \Gamma(s)(k/2\pi)^s k^{-1} G(\chi)\, e^{-2\pi iax/k} \cdot \{e^{-\pi is/2} L(s, a, -x, \bar{\chi}) + \chi(-1)\, e^{\pi is/2} L(s, -a, x, \bar{\chi})\}. \tag{5.8}$$

Proof. We shall always choose that branch of $\log z$ where $-\pi < \arg z \leqslant \pi$.

First suppose that x is an integer. In this case, a proof of (5.8) is easily derived from Hurwitz's formula. Let $n = mk + j$, $0 \leqslant m < \infty$, $0 \leqslant j \leqslant k-1$. Then for $\sigma > 1$,

$$L(s, x, a, \chi) = \sum_{n=0}^{\infty} e^{2\pi inx/k}\chi(n)(n+a)^{-s}$$

$$= k^{-s} \sum_{j=1}^{k-1} \chi(j)\, e^{2\pi ijx/k}\zeta(s, (j+a)/k),$$

since $0 < (j+a)/k \leqslant 1$. Since $\zeta(s, (j+a)/k)$ can be analytically continued to a function analytic everywhere except for a simple pole at $s = 1$ with residue 1, the statements in Theorem 5.1 about analytic continuation follow immediately with the use of (2.1).

Replacing s by $1 - s$ in the above and using Hurwitz's formula [24, p. 39], we have for $\sigma > 1$,

$$L(1-s, x, a, \chi) = k^{s-1} \sum_{j=1}^{k-1} \chi(j)\, e^{2\pi ijx/k} (2\pi)^{-s}\, \Gamma(s)$$

$$\cdot \left\{ e^{-\pi is/2} \sum e^{2\pi in(j+a)/k} n^{-s} + e^{\pi is/2} \sum e^{-2\pi in(j+a)/k} n^{-s} \right\}$$

$$= (k/2\pi)^s\, k^{-1} \Gamma(s) \left\{ e^{-\pi is/2} \sum e^{2\pi ina/k} G(n+x, \chi)\, n^{-s} \right.$$

$$\left. + e^{\pi is/2} \chi(-1) \sum e^{-2\pi ina/k} G(n-x, \chi)\, n^{-s} \right\}, \qquad (5.9)$$

upon inverting the orders of summation. Using (2.1) and replacing $n + x$ by n and $n - x$ by n, respectively, in the two sums on the far right side of (5.9), we get for $\sigma > 1$,

$$L(1-s, x, a, \chi)$$

$$= (k/2\pi)^s\, k^{-1} \Gamma(s)\, G(\chi) e^{-2\pi iax/k} \left\{ e^{-\pi is/2} \sum_{n=x+1}^{\infty} e^{2\pi ina/k} \bar{\chi}(n)(n-x)^{-s} \right.$$

$$\left. + e^{\pi is/2} \chi(-1) \sum_{n=1-x}^{\infty} e^{-2\pi ina/k} \bar{\chi}(n)(n+x)^{-s} \right\}. \qquad (5.10)$$

Replacing n by $-n$, we observe that if $x \geqslant 0$,

$$e^{\pi is/2} \chi(-1) \sum_{n=1-x}^{0} e^{-2\pi ina/k} \bar{\chi}(n)(n+x)^{-s} = e^{-\pi is/2} \sum_{n=0}^{x-1} e^{2\pi ina/k} \bar{\chi}(n)(n-x)^{-s}. \qquad (5.11)$$

Substituting (5.11) into (5.10), we immediately arrive at (5.8).

Assume now that $0 < x < 1$ and that $a > 0$. (The proof for $a = 0$ requires but simple modifications.) Suppose that χ is odd. Let $f(u) = (u + a)^{-s} \exp(2\pi iux/k)$, $\sigma > 0$, and apply (4.2) with $b = Nk$, where N is a positive integer, to obtain

$$\sum_{1-a \leqslant n \leqslant Nk}' \chi(n) f(n) = B_1(\bar{\chi})\{f(Nk) - f(1-a)\} - \int_{1-a}^{Nk} \mathscr{B}_1(u, \bar{\chi}) f'(u)\, du,$$

since $\mathscr{B}_1(u, \bar{\chi})$ is constant on $(-1, 1)$. Letting N tend to ∞, we obtain for $\sigma > 0$,

$$L(s, x, a, \chi) = -B_1(\bar{\chi}) f(1-a) + s \int_{1-a}^{\infty} \mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k} (u+a)^{-s-1}\, du$$

$$- (2\pi ix/k) \int_{1-a}^{\infty} \mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k} (u+a)^{-s}\, du.$$

By an application of Dirichlet's test, the integrals above are seen to converge for $\sigma > 0$.

Now assume that $0 < \sigma < 1$. Using Definition 1 and inverting the order of summation and integration, we find that

$$\int_{-a}^{\infty} \frac{\mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k}}{(u+a)^s}\, du$$
$$= \frac{iG(\chi)}{\pi} \sum \frac{\bar{\chi}(n)}{n} \int_{-a}^{\infty} \frac{\cos(2\pi nu/k)\, e^{2\pi iux/k}}{(u+a)^s}\, du$$
$$= \frac{iG(\chi)\, e^{-2\pi iax/k}}{2\pi} \sum \frac{\bar{\chi}(n)}{n} \left\{ e^{-2\pi ina/k} \int_0^{\infty} e^{2\pi iu(x+n)/k} u^{-s}\, du \right.$$
$$\left. + e^{2\pi ina/k} \int_0^{\infty} e^{2\pi iu(x-n)/k} u^{-s}\, du \right\},$$

where we have replaced u by $u - a$. Now, if $0 < \sigma < 1$ [23, pp. 107–108],

$$\int_0^{\infty} u^{-s} e^{idu}\, du = \Gamma(1-s)\, |\, d\, |^{s-1}\, e^{\frac{1}{2}\pi i(1-s)\operatorname{sgn} d}.$$

Thus,

$$\int_{-a}^{\infty} \frac{\mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k}}{(u+a)^s}\, du$$
$$= \frac{iG(\chi)\, e^{-2\pi iax/k}\Gamma(1-s)}{2\pi} \sum \frac{\bar{\chi}(n)}{n} \left\{ \left(\frac{2\pi(x+n)}{k} \right)^{s-1} e^{-2\pi ina/k + \frac{1}{2}\pi i(1-s)} \right.$$
$$\left. + \left(\frac{2\pi\, |\, x-n\, |}{k} \right)^{s-1} e^{2\pi ina/k - \frac{1}{2}\pi i(1-s)} \right\}. \tag{5.12}$$

Thus, for $0 < \sigma < 1$,

$$L(s, x, a, \chi)$$
$$= -B_1(\bar{\chi}) f(1-a) + s \int_{1-a}^{\infty} \mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k}(u+a)^{-s-1}\, du$$
$$+ (2\pi ix/k) \int_{-a}^{1-a} \mathscr{B}_1(u, \bar{\chi})\, e^{2\pi iux/k}(u+a)^{-s-1}\, du$$
$$+ \Gamma(1-s)(2\pi/k)^{s-1}\, k^{-1} x G(\chi)\, e^{-2\pi iax/k} \sum \frac{\bar{\chi}(n)}{n}$$
$$\cdot \{(x+n)^{s-1}\, e^{-2\pi ina/k + \frac{1}{2}\pi i(1-s)} + |\, x-n\, |^{s-1}\, e^{2\pi ina/k - \frac{1}{2}\pi i(1-s)}\}. \tag{5.13}$$

The first integral on the right side of (5.13) converges uniformly for $\sigma \geqslant -1 + \epsilon$, $\epsilon > 0$, by Dirichlet's test. Hence, (5.13) gives an analytic continuation for $L(s, x, a, \chi)$ in the strip $-1 < \sigma < 1$.

Now assume that $-1 < \sigma < 0$. Replacing s by $s+1$ in (5.12), we obtain

$$\int_{-a}^{\infty} \frac{\mathscr{B}_1(u, \bar{\chi})\, e^{2\pi i u x/k}}{(u+a)^{s+1}}\, du$$
$$= \frac{iG(\chi)\, e^{-2\pi i a x/k}\Gamma(-s)(2\pi/k)^s}{2\pi}$$
$$\cdot \sum \frac{\bar{\chi}(n)}{n} \{(x+n)^s\, e^{-2\pi i n a/k - \frac{1}{2}\pi i s} + |x-n|^s\, e^{2\pi i n a/k + \frac{1}{2}\pi i s}\}. \quad (5.14)$$

Next, since $\mathscr{B}_1(u, \bar{\chi})$ is constant on $(-1, 1)$,

$$-s \int_{-a}^{1-a} \frac{\mathscr{B}_1(u, \bar{\chi})\, e^{2\pi i u x/k}}{(u+a)^{s+1}}\, du + \frac{2\pi i x}{k} \int_{-a}^{1-a} \frac{\mathscr{B}_1(u, \bar{\chi})\, e^{2\pi i u x/k}}{(u+a)^{s}}\, du$$
$$= B_1(\bar{\chi}) \int_{-a}^{1-a} f'(u)\, du = B_1(\bar{\chi}) f(1-a). \quad (5.15)$$

Put (5.14) and (5.15) into (5.13) and use the facts that $x/\{n(x+n)\} - 1/n = -1/(n+x)$ and

$$\frac{xe^{-\frac{1}{2}\pi i(1-s)}}{|x-n|^{1-s}} - \frac{ie^{\frac{1}{2}\pi i s}}{|x-n|^{-s}} = -\frac{nie^{\frac{1}{2}\pi i s}}{(n-x)^{1-s}}.$$

Upon simplification, we arrive at (5.8) but with s replaced by $1-s$. By analytic continuation, (5.8) holds for all s.

The proof of (5.8) when χ is even follows along the same lines, but, of course, uses (4.1) instead of (4.2). The proof is actually slightly simpler because for even χ, $\mathscr{B}_1(u, \bar{\chi}) \equiv 0$ on $(-1, 1)$.

If $x = 0$ and χ is even, (5.8) reduces to

$$L(1-s, 0, a, \chi) = 2\Gamma(s)(k/2\pi)^s\, k^{-1}G(\chi) \sum \bar{\chi}(n) \cos(2\pi na/k - \pi s/2)\, n^{-s}; \quad (5.16)$$

if $x = 0$ and χ is odd, (5.8) reduces to

$$L(1-s, 0, a, \chi) = 2i\Gamma(s)(k/2\pi)^s\, k^{-1}G(\chi) \sum \bar{\chi}(n) \sin(2\pi na/k - \pi s/2)\, n^{-s}. \quad (5.17)$$

If we let $s = j$ be a positive integer, both (5.16) and (5.17) give by Definition 1, for $0 \leqslant a < 1$,

$$L(1-j, 0, a, \chi) = -B_j(a, \bar{\chi})/j,$$

which is analogous to the familiar formula $\zeta(1-j, a) = -B_j(a)/j$. Observe that if $a = 0$, the above reduces to

$$L(1-j, \chi) = -B_j(\bar{\chi})/j.$$

EXAMPLE 5. We would like to now show a connection between generalized Bernoulli functions and some identities for logarithmic means that we previously derived [7, 8].

PROPOSITION 5.2. *As x tends to ∞, we have the following asymptotic expansion,*

$$\sum_{n \leqslant x} \chi(n) \log(x/n) \sim L(0, \chi) \log x + L'(0, \chi) - \chi(-1) \sum_{j=2}^{\infty} \frac{\mathscr{B}_j(x, \bar{\chi})}{j(j-1)\, x^{j-1}}. \tag{5.18}$$

Proof. First, suppose that χ is even. Using (2.2), apply Theorem 2 of [7] to obtain

$$\sum_{\pi n^2/k \leqslant x} \chi(n) \log(xk/\pi n^2)$$

$$= R(x; 1) - 2^{1/2}\epsilon(\chi) \sum \frac{\bar{\chi}(n)}{(\pi n^2/k)^{1/2}} \int_{2(\pi n^2 x/k)^{1/2}}^{\infty} u^{-1/2} J_{1/2}(u)\, du, \tag{5.19}$$

where $R(x; 1)$ is the residue of $(\pi/k)^{-s} L(2s, \chi)\, x^s/s^2$ at $s = 0$. Replace x by $\pi x^2/k$, use the relation $J_{1/2}(u) = (2/\pi u)^{1/2} \sin u$ and note that $L(0, \chi) = 0$. Equation (5.19) then becomes

$$\sum_{n \leqslant x} \chi(n) \log(x/n)$$

$$= L'(0, \chi) - \frac{G(\chi)}{\pi} \sum \frac{\bar{\chi}(n)}{n} \int_{2\pi n x/k}^{\infty} \frac{\sin u}{u}\, du$$

$$\sim L'(0, \chi) - \frac{G(\chi)}{\pi} \sum \frac{\bar{\chi}(n)}{n} \left\{ \frac{\cos(2\pi n x/k)}{2\pi n x/k} + \frac{\sin(2\pi n x/k)}{(2\pi n x/k)^2} \right.$$

$$\left. - \frac{2!\cos(2\pi n x/k)}{(2\pi n x/k)^3} - \frac{3!\sin(2\pi n x/k)}{(2\pi n x/k)^4} + \cdots \right\}, \tag{5.20}$$

where we have integrated by parts repeatedly. If we now use Definition 1, we arrive at (5.18) at once.

Next let χ be odd. From [8, p. 346],

$$\sum_{n \leqslant x} \chi(n) \log(x/n)$$

$$= L(0, \chi) \log x + L'(0, \chi) + \frac{k^{1/2}\epsilon(\chi)}{\pi} \sum \frac{\bar{\chi}(n)}{n} \int_{2\pi n x/k}^{\infty} \frac{\cos u}{u}\, du.$$

(In [8] a factor of $1/\pi$ has been omitted from lines 7 and 10 from the bottom of p. 346.) If we repeatedly integrate by parts as before and use Definition 1 again, we obtain (5.18) immediately.

If $x = N$, a positive integer, (5.18) has a striking resemblance to Stirling's formula for $\log N!$ (In particular, see [7, p. 371] where Stirling's formula is shown to be a consequence of the functional equation of the Riemann zeta-function.) This motivates us to define the following character analogue of the gamma function.

DEFINITION 4. Let χ be a real primitive character. Define

$$\Gamma(s, \chi) = \prod_{n=1}^{\infty} \left(\frac{n}{n+s}\right)^{\chi(n)}. \tag{5.21}$$

That $\Gamma(s, \chi)$ is well defined will be seen from considerations below.

PROPOSITION 5.3 (Stirling's formula for $\log \Gamma(s, \chi)$). *For $-\pi < \arg s < \pi$, as x tends to ∞,*

$$\log \Gamma(s, \chi) \sim -L(0, \chi) \log s - L'(0, \chi) + \chi(-1) \sum_{j=2}^{\infty} \frac{B_j(\chi)}{j(j-1)\, s^{j-1}}, \tag{5.22}$$

where the principal branch of the logarithm is taken.

Proof. Let χ be even and let N be a positive integer. Integrating by parts twice in (5.20) and replacing u by $2\pi nu/k$, we arrive at

$$\begin{aligned} -\sum_{n \leqslant Nk} \chi(n) \log n &= \sum_{n \leqslant Nk} \chi(n) \log(Nk/n) \\ &= L'(0, \chi) - \frac{G(\chi)\, L(2, \chi)}{2\pi^2 N} + \frac{1}{3} \int_{Nk}^{\infty} \frac{\mathscr{B}_3(u, \chi)}{u^3}\, du. \end{aligned} \tag{5.23}$$

Next apply (4.3) with $f(u) = \log(u + s)$, $-\pi < \arg s < \pi$, $A = 0$, $B = N$, and $r = 1$ to get

$$\sum_{n \leqslant Nk} \chi(n) \log(n + s) = \frac{B_2(\chi)}{2} \left\{\frac{1}{Nk + s} - \frac{1}{s}\right\} + \frac{1}{3} \int_0^{Nk} \frac{\mathscr{B}_3(u, \chi)}{(u + s)^3}\, du. \tag{5.24}$$

Combining (5.23) and (5.24), we have

$$\begin{aligned} \log \Gamma(s, \chi) &= \lim_{N \to \infty} \sum_{n \leqslant Nk} \chi(n) \{\log n - \log(n + s)\} \\ &= -L'(0, \chi) + \frac{B_2(\chi)}{2s} - \frac{1}{3} \int_0^{\infty} \frac{\mathscr{B}_3(u, \chi)}{(u + s)^3}\, du. \end{aligned}$$

This shows that $\Gamma(s, \chi)$ is well defined and analytic for $-\pi < \arg s < \pi$.
In a similar fashion, it can be shown that if χ is odd and $-\pi < \arg s < \pi$,

$$\log \Gamma(s, \chi) = -L(0, \chi) \log s - L'(0, \chi) + \frac{1}{2} \int_0^\infty \frac{\mathscr{B}_2(u, \chi)}{(u+s)^2}\, du.$$

The asymptotic formula (5.22) now follows in both cases by integrating by parts repeatedly with the help of Corollary 3.3.

PROPOSITION 5.4 (Analogue of the Weierstrass product representation of $\Gamma(s)$). *We have for all s*

$$\Gamma(s, \chi) = e^{-sL(1,\chi)} \prod_{n=1}^{\infty} (1 + s/n)^{-\chi(n)}\, e^{s\chi(n)/n}, \tag{5.25}$$

where the product converges uniformly on any compact set S which avoids the points $s = -n$, where n is a positive integer and $\chi(n) = 1$.

Proof. From (5.21),

$$\begin{aligned} \Gamma(s, \chi) &= \lim_{N\to\infty} \prod_{n=1}^{N} (1 + s/n)^{-\chi(n)} \\ &= \lim_{N\to\infty} \exp\left(-s \sum_{j=1}^{N} \chi(j)/j\right) \prod_{n=1}^{N} (1 + s/n)^{-\chi(n)}\, e^{s\chi(n)/n} \\ &= e^{-sL(1,\chi)} \prod_{n=1}^{\infty} (1 + s/n)^{-\chi(n)}\, e^{s\chi(n)/n}. \end{aligned}$$

If $s \in S$,

$$(1 + s/n)^{-\chi(n)}\, e^{s\chi(n)/n} = 1 - s^2/n^2 + O(n^{-2}) = 1 + O(n^{-2}),$$

where the constant implied by the big O is independent of $s \in S$. This shows that the product is uniformly convergent on S.

We see from Proposition 5.4 that $\Gamma(s, \chi)$ is analytic everywhere except at the points $s = -n$, where n is a positive integer and $\chi(n) = 1$, in which cases there are simple poles. If $\chi(n) = -1$, $n > 0$, $\Gamma(s, \chi)$ has a simple zero at $s = -n$. These are the only zeros of $\Gamma(s, \chi)$.

We will next show, in fact, that $\Gamma(s, \chi)$ is a quotient of ordinary gamma functions. To do this we need the following lemma.

LEMMA 5.5. *Define for $0 < a \leqslant 1$,*

$$\gamma(a) = \lim_{N\to\infty} \left\{ \sum_{n=0}^{N} (n + a)^{-1} - \log(N + a) \right\}. \tag{5.26}$$

Then,

$$L(1, \chi) = \frac{1}{k} \sum_{n=1}^{k-1} \chi(n)\, \gamma(n/k).$$

Proof. The limit in (5.26) is easily seen to exist by a routine application of the ordinary Euler–Maclaurin sum formula. Note that $\gamma(1) = \gamma$, Euler's constant.

Letting $j = nk + r$ below, we have

$$\begin{aligned}
\frac{1}{k} \sum_{r=1}^{k-1} \chi(r)\, \gamma(r/k) &= \lim_{N\to\infty} \sum_{r=1}^{k-1} \chi(r) \left\{ \sum_{n=0}^{N} (nk + r)^{-1} - \frac{1}{k} \log(N + r/k) \right\} \\
&= \lim_{N\to\infty} \left\{ \sum_{j=0}^{Nk+k-1} \chi(j)/j - \frac{1}{k} \sum_{r=1}^{k-1} \chi(r) \log(N + r/k) \right\} \\
&= L(1, \chi) - \lim_{N\to\infty} \frac{1}{k} \sum_{r=1}^{k-1} \chi(r) \{\log N + \log(1 + r/Nk)\} \\
&= L(1, \chi).
\end{aligned}$$

PROPOSITION 5.6. *We have for all s,*

$$\Gamma(s, \chi) = \prod_{r=1}^{k-1} \left\{ \frac{\Gamma(\{s + r\}/k)}{\Gamma(r/k)} \right\}^{\chi(r)}.$$

Proof. In (5.25) put $n = mk + r$, $0 \leqslant m < \infty$, $0 \leqslant r \leqslant k - 1$. Using Lemma 5.5, we then have

$$\begin{aligned}
\Gamma(s, \chi) &= \exp\left(-\frac{s}{k} \sum_{n=1}^{k-1} \chi(n)\, \gamma(n/k)\right) \prod_{r=0}^{k-1} \left\{ \prod_{m=0}^{\infty} \left(1 + \frac{s/k}{m + r/k}\right)^{-1} e^{\frac{s/k}{m+r/k}} \right\}^{\chi(r)} \\
&= \prod_{r=1}^{k-1} \left\{ e^{-s\gamma(r/k)/k} \prod_{m=0}^{\infty} \left(1 + \frac{s/k}{m + r/k}\right)^{-1} e^{\frac{s/k}{m+r/k}} \right\}^{\chi(r)} \\
&= \prod_{r=1}^{k-1} \Gamma_{r/k}(s/k)^{\chi(r)}, \qquad (5.27)
\end{aligned}$$

where for $0 < a \leqslant 1$,

$$\begin{aligned}
\Gamma_a(s) &= e^{-\gamma(a)s} \prod_{m=0}^{\infty} \left(1 + \frac{s}{m + a}\right)^{-1} e^{s/(m+a)} \\
&= \lim_{N\to\infty} \exp\left\{ -s \left(\sum_{n=0}^{N} (n + a)^{-1} - \log(N + a) \right) \right\}
\end{aligned}$$

$$\times \prod_{m=0}^{N} \left(1 + \frac{s}{m+a}\right)^{-1} e^{s/(m+a)}$$

$$= \lim_{N\to\infty} e^{s \log(N+a)} \prod_{m=0}^{N} \left(1 + \frac{s}{m+a}\right)^{-1}$$

$$= \lim_{N\to\infty} \frac{(N+a)^s \, a(a+1) \cdots (a+N)}{(a+s)(a+1+s) \cdots (a+N+s)}$$

$$= \lim_{N\to\infty} \frac{N! \, N^{s+a} a(a+1) \cdots (a+N)}{(a+s)(a+1+s) \cdots (a+N+s) \, N! \, N^a}$$

$$= \frac{\Gamma(s+a)}{\Gamma(a)},$$

upon the use of the familiar product representation of $\Gamma(s)$ [5, p. 354]. Upon the substitution of the above in (5.27), the proof of Proposition 5.6 is complete.

EXAMPLE 6. First, from Definition 1, we obtain the trivial estimate

$$|\mathscr{B}_j(x, \chi)| \leqslant \frac{2\,|G(\bar{\chi})|\, j!\, \zeta(j)}{k(2\pi/k)^j}. \tag{5.28}$$

Put $f(x) = e^{xy}$, $A = 0$, and $B = 1$ in (4.3). Then for even χ,

$$\sum_{n=1}^{k-1} \chi(n)\, e^{ny} = \sum_{j=1}^{r} \frac{B_{2j}(\bar{\chi})}{(2j)!} y^{2j-1}(e^{ky} - 1) + E_r,$$

where

$$|E_r| \leqslant \frac{1}{(2r+1)!} \int_0^k |\mathscr{B}_{2r+1}(u, \bar{\chi})|\, y^{2r+1} e^{uy}\, du$$
$$\leqslant 2\,|G(\bar{\chi})|\, e^{ky} \zeta(2r+1)(k\,|y|/2\pi)^{2r+1},$$

upon the use of (5.28). Hence, if $|y| < 2\pi/k$, E_r tends to 0 as r tends to ∞. Thus, for $|y| < 2\pi/k$,

$$\sum_{n=1}^{k-1} \chi(n)\, e^{ny} = \sum \frac{B_{2j}(\bar{\chi})}{(2j)!} y^{2j-1}(e^{ky} - 1),$$

or

$$\frac{y \sum_{n=1}^{k-1} \chi(n)\, e^{ny}}{e^{ky} - 1} = \sum \frac{B_j(\bar{\chi})\, y^j}{j!}. \tag{5.29}$$

If χ is odd, an analogous calculation gives (5.29) as well.

Ankeny, Artin and Chowla [2] and Leopoldt [20] defined generalized Bernoulli numbers for real primitive characters by means of (5.29). Thus, we have shown that our definition of generalized Bernoulli numbers is equivalent to previous definitions. In fact, Theorem 4.2 or Eqs. (4.7) and (4.8) show this as well.

EXAMPLE 7. Let χ be even. In (4.3) put $f(x) = \cos(xy)$, $A = 0$, and $B = 1$. Then,

$$\sum_{n=1}^{k-1} \chi(n)\cos(ny) = \sum_{j=1}^{r} \frac{B_{2j}(\bar{\chi})}{(2j)!}(-1)^j y^{2j-1}\sin(ky) + E_r,$$

where by a calculation similar to that in Example 6, E_r tends to 0 as r tends to ∞ if $|y| < 2\pi/k$. Thus, for $|y| < 2\pi/k$,

$$\frac{y\sum_{n=1}^{k-1}\chi(n)\cos(ny)}{\sin(ky)} = \sum \frac{B_{2j}(\bar{\chi})(-1)^j y^{2j}}{(2j)!}. \tag{5.30}$$

Equation (5.30) may be regarded as the natural generalization of the well-known Laurent expansion of $\cot y$ in terms of the ordinary Bernoulli numbers.

Let χ be odd. In (4.4) put $f(x) = \cos(xy)$, $A = 0$, and $B = 1$. By an argument similar to that above, we have for $|y| < 2\pi/k$,

$$\frac{y\sum_{n=1}^{k-1}\chi(n)\cos(ny)}{\cos(ky)-1} = \sum \frac{B_{2j-1}(\bar{\chi})(-1)^{j-1} y^{2j-1}}{(2j-1)!}. \tag{5.31}$$

EXAMPLE 8. Let χ be even. Put $f(x) = \sin(xy)$, $A = 0$, and $B = 1$ in (4.3). By the same manner of reasoning as in Example 7, we have for $|y| < 2\pi/k$,

$$\frac{y\sum_{n=1}^{k-1}\chi(n)\sin(ny)}{\cos(ky)-1} = \sum \frac{B_{2j}(\bar{\chi})(-1)^{j-1} y^{2j}}{(2j)!}. \tag{5.32}$$

Similarly, if χ is odd, we find from (4.4) that for $|y| < 2\pi/k$,

$$\frac{y\sum_{n=1}^{k-1}\chi(n)\sin(ny)}{\sin(ky)} = \sum \frac{B_{2j-1}(\bar{\chi})(-1)^{j-1} y^{2j-1}}{(2j-1)!}. \tag{5.33}$$

By comparing (5.30) and (5.32), we deduce that for even χ,

$$\frac{\sum_{n=1}^{k-1}\chi(n)\cos(ny)}{\sin(ky)} = \frac{-\sum_{n=1}^{k-1}\chi(n)\sin(ny)}{\cos(ky)-1}. \tag{5.34}$$

By comparing (5.31) and (5.33), we find that for odd χ,

$$\frac{\sum_{n=1}^{k-1} \chi(n) \cos(ny)}{\cos(ky) - 1} = \frac{\sum_{n=1}^{k-1} \chi(n) \sin(ny)}{\sin(ky)}. \tag{5.35}$$

The two infinite classes of trigonometric identities (5.34) and (5.35) are not as deep as they might appear at first glance. Cross-multiplying and simplifying, we easily deduce that (5.34) is equivalent to

$$\sum_{n=1}^{k-1} \chi(n) \cos(ny) = \sum_{n=1}^{k-1} \chi(n) \cos(n-k)\, y. \tag{5.36}$$

But (5.36) can be trivially established for all complex y and any even character by replacing n be $k - n$ in the left side of (5.36). Thus, in fact, (5.34) holds for any even character. Similarly, (5.35) is equivalent to

$$\sum_{n=1}^{k-1} \chi(n) \sin(ny) = \sum_{n=1}^{k-1} \chi(n) \sin(n-k)\, y,$$

which can be proved for any odd character and all complex y. Thus, (5.35) is valid for any odd character.

EXAMPLE 9. Let $f(x) = \cosh(xy)$, $A = 0$, and $B = 1$. If χ is even, we obtain from (4.3) for $|y| < 2\pi/k$,

$$\frac{y \sum_{n=1}^{k-1} \chi(n) \cosh(ny)}{\sinh(ky)} = \sum \frac{B_{2j}(\bar{\chi})\, y^{2j}}{(2j)!}. \tag{5.37}$$

If χ is odd, (4.4) yields for $|y| < 2\pi/k$,

$$\frac{y \sum_{n=1}^{k-1} \chi(n) \cosh(ny)}{\cosh(ky) - 1} = \sum \frac{B_{2j-1}(\bar{\chi})\, y^{2j-1}}{(2j-1)!}. \tag{5.38}$$

We may regard (5.37) as the natural generalization of the Laurent expansion of $\coth y$ in terms of the ordinary Bernoulli numbers.

EXAMPLE 10. Let $f(x) = \sinh(xy)$, $A = 0$, and $B = 1$. If χ is even, (4.3) gives for $|y| < 2\pi/k$,

$$\frac{y \sum_{n=1}^{k-1} \chi(n) \sinh(ny)}{\cosh(ky) - 1} = \sum \frac{B_{2j}(\bar{\chi})\, y^{2j}}{(2j)!}. \tag{5.39}$$

If χ is odd, (4.4) yields for $|y| < 2\pi/k$,

$$\frac{y \sum_{n=1}^{k-1} \chi(n) \sinh(ny)}{\sinh(ky)} = \sum \frac{B_{2j-1}(\bar{\chi})\, y^{2j-1}}{(2j-1)!}. \tag{5.40}$$

By comparing (5.37) and (5.39), we find that for χ even,

$$\frac{\sum_{n=1}^{k-1} \chi(n) \cosh(ny)}{\sinh(ky)} = \frac{\sum_{n=1}^{k-1} \chi(n) \sinh(ny)}{\cosh(ky) - 1}.$$

Comparing (5.38) and (5.40), we have for odd χ,

$$\frac{\sum_{n=1}^{k-1} \chi(n) \cosh(ny)}{\cosh(ky) - 1} = \frac{\sum_{n=1}^{k-1} \chi(n) \sinh(ny)}{\sinh(ky)}.$$

The above two infinite classes of hyperbolic trigonometric identities, in fact, hold for any even or odd character, respectively, by an argument similar to that made above.

6. Further Properties of Generalized Bernoulli Functions, Polynomials and Numbers

L. Carlitz [11] has given several properties of generalized Bernoulli numbers and polynomials. We give a few more properties here.

A well-known property of Bernoulli polynomials is [1, p. 804]

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j}. \tag{6.1}$$

We now derive the analogue for generalized Bernoulli polynomials.

Proposition 6.1. *For $n \geq 1$,*

$$B_n(x, \chi) = \sum_{j=0}^{n} \binom{n}{j} B_j(\chi)\, x^{n-j}.$$

Proof. From a slightly more general form of (6.1) [1, p. 804],

$$B_n(x + h) = \sum_{j=0}^{n} \binom{n}{j} B_j(h)\, x^{n-j},$$

and Corollary 3.2, we have

$$\begin{aligned} B_n(x, \chi) &= k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) \sum_{j=0}^{n} \binom{n}{j} B_j(h/k)(x/k)^{n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} k^{j-1} \sum_{h=1}^{k-1} \bar{\chi}(h)\, B_j(h/k) \\ &= \sum_{j=0}^{n} \binom{n}{j} B_j(\chi)\, x^{n-j}. \end{aligned}$$

PROPOSITION 6.2. *For* $|\, y \,| < 2\pi/k$,

$$\frac{y \sum_{n=1}^{k-1} \chi(n)\, e^{(n+x)y}}{e^{ky} - 1} = \sum \frac{B_j(x, \bar{\chi})\, y^j}{j!}. \tag{6.2}$$

Proof. Multiply both sides of (5.6) by e^{xy}. If we use the Maclaurin series for e^{xy} and Proposition 6.1, the result easily follows.

Leopoldt [20] defined the generalized Bernoulli polynomials for real primitive characters by (6.2). Thus, our definition is equivalent to his.

The next result is analogous to the well-known fact [1, p. 804],

$$B_n(x + 1) - B_n(x) = nx^{n-1}. \tag{6.3}$$

PROPOSITION 6.3. *We have*

$$B_n(x + k, \chi) - B_n(x, \chi) = n \sum_{h=1}^{k-1} \bar{\chi}(h)(x + h)^{n-1}.$$

Proof. By Corollary 3.2 and (6.3),

$$\begin{aligned} &B_n(x + k, \chi) - B_n(x, \chi) \\ &\quad = k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) \left\{ B_n\left(\frac{x+h}{k} + 1\right) - B_n\left(\frac{x+h}{k}\right) \right\} \\ &\quad = k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h)\, n \left(\frac{x+h}{k}\right)^{n-1}, \end{aligned}$$

and the result follows.

In contrast to the ordinary Bernoulli polynomials, $B_n(x, \chi)$ is either an even or odd function as the next result shows.

PROPOSITION 6.4. *We have*

$$B_n(-x, \chi) = (-1)^{n+b} B_n(x, \chi).$$

Proof. Using Corollary 3.2 and replacing h by $k - h$, we obtain

$$\begin{aligned} B_n(-x, \chi) &= k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) B_n\left(\frac{-x+h}{k}\right) \\ &= (-1)^b k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) B_n\left(\frac{-x-h}{k} + 1\right). \end{aligned}$$

Since [1, p. 804]

$$B_n(1 - x) = (-1)^n B_n(x),$$

$$B_n(-x, \chi) = (-1)^{n+b} k^{n-1} \sum_{h=1}^{k-1} \bar{\chi}(h) B_n\left(\frac{x+h}{k}\right) = (-1)^{n+b} B_n(x, \chi).$$

Since the degree of $B_n(x, \chi)$ is no greater than $n - 1$, we see from Theorem 6.5 that if χ is even, the degree of $B_n(x, \chi)$, $n \geqslant 2$, does not exceed $n - 2$. Also note that if χ is even, $B_1(x, \chi) \equiv 0$.

The following theorem gives formulas that are analogous to a familiar formula for $\zeta(2j - 1)$ [1, p. 807].

PROPOSITION 6.5. *Let $j \geqslant 1$. If χ is even,*

$$L(2j - 1, \chi) = \frac{(-1)^j (2\pi/k)^{2j-1}}{2G(\bar{\chi})(2j-1)!} \int_0^k \mathscr{B}_{2j-1}(u, \chi) \cot(\pi u/k)\, du;$$

if χ is odd,

$$L(2j, \chi) = \frac{(-1)^j\, i(2\pi/k)^{2j}}{2G(\bar{\chi})(2j)!} \int_0^k \mathscr{B}_{2j}(u, \chi) \cot(\pi u/k)\, du.$$

Proof. Let χ be even. If n is a positive integer [14, p. 366],

$$I_n = \int_0^k \sin(2\pi n u/k) \cot(\pi u/k)\, du = k.$$

Using the above and Definition 1, we have

$$\begin{aligned} \int_0^k \mathscr{B}_{2j-1}(u, \chi) \cot(\pi u/k)\, du &= \frac{2(-1)^j G(\bar{\chi})(2j-1)!}{k(2\pi/k)^{2j-1}} \sum \frac{\chi(n)}{n^{2j-1}} I_n \\ &= \frac{2(-1)^j G(\bar{\chi})(2j-1)!}{(2\pi/k)^{2j-1}} L(2j - 1, \chi), \end{aligned}$$

and the result follows.

The proof for odd χ is analogous.

PROPOSITION 6.6. *For $m, n \geqslant 1$,*

$$\int_0^k \mathscr{B}_m(u, \chi)\, \mathscr{B}_n(u, \bar{\chi})\, du = \frac{(-1)^{m-1} k^{m+n} m!\, n!\, B_{m+n}}{(m+n)!}.$$

Proof. Let χ be even, and assume first that $m = 2j - 1$ and $n = 2l - 1$ are odd. Using Definition 1 and (4.9), we have

$$\begin{aligned}
&\int_0^k \mathscr{B}_m(u, \chi)\, \mathscr{B}_n(u, \bar{\chi})\, du \\
&= \frac{4(-1)^{j+l} m!\, n!}{k(2\pi/k)^{m+n}} \sum \frac{\chi(n_1)}{n_1{}^m} \sum \frac{\bar{\chi}(n_2)}{n_2{}^n} \int_0^k \sin(2\pi n_1 u/k) \sin(2\pi n_2 u/k)\, du \\
&= \frac{2(-1)^{j+l} m!\, n!}{(2\pi/k)^{m+n}} \zeta(m+n) \\
&= \frac{k^{m+n} m!\, n!\, B_{m+n}}{(m+n)!},
\end{aligned}$$

where we have used the well-known fact that [1, p. 807]

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!} B_{2n},$$

for any positive integer n.

The proofs for the remaining parities of m and n are similar. Also, the proof for odd χ is analogous.

PROPOSITION 6.7. *For $m, n \geqslant 1$,*

$$\int_0^1 B_m(ku, \chi)\, B_n(u)\, du = \frac{(-1)^{m-1+b} m!\, n!\, B_{m+n}(\chi)}{k^n (m+n)!}.$$

Proof. The proof is similar to the previous proof and uses Definition 1, and Eqs. (3.1), (3.2), (4.7) and (4.8).

Note added in proof. Some of the material in this paper can be found in a more general setting in a paper coauthored with Lowell Schoenfeld, Periodic analogues of the Euler-MacLaurin and Poisson summation formulas with applications to number theory, which is in the course of publication in *Acta Arithmetica.* For several additional applications of the character Poisson formula, see the author's paper, Periodic Bernoulli numbers, summation formulas and applications, theory and applications of special functions, Academic Press, New York, 1975.

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